The Mathematics of Cognitively Guided Instruction

Grade 1

Mathematics has been -- is -- and will continue to be -- a major part of the elementary school curriculum. Starting in Kindergarten and continuing through the last year of elementary school, the teaching of mathematics is second in importance only to the teaching of reading. Not only is there consensus about the importance of mathematics, there is also an unwritten consensus about the mathematics content emphasis of each grade. While there is some overlap, basically addition and subtraction are emphasized in the first and second grades; multiplication and division in the third and fourth grades; fractions and decimals in the fifth and sixth grades. Some geometric and measurement ideas are included each year also, but what is taught in these two areas varies depending upon the teacher, textbook and school. While we recognize the importance of the other content, this project is focused on addition and subtraction in the first grade.

The idea that addition and subtraction is the focus of mathematics instruction in Grade 1 is not particularly helpful in deciding specifically what should be taught. For example, one could decide the emphasis should be on understanding addition's place in the structure of mathematics. If this decision is made, then certain ideas would be used as the focus of instruction. E.g., subtraction might be taught as a special form of addition, the commutative and associative properties of addition would be emphasized, and zero would be taught as the identity element. Instead of deciding to focus on the structure of mathematics, one could decide that addition should be mastered before a child could proceed to understand any advanced mathematics or to use addition. This focus would suggest that children should memorize the basic facts as quickly as possible, learn to add two digit numbers with and without carrying and then to practice these skills until a high degree of accuracy is obtained. Similar activities would be done with subtraction.

We do not believe that instruction focused in either of these ways increases children's understanding of the structure of mathematics, or results in children being able to use addition and subtraction in problem solving. We believe that the focus of addition and subtraction instruction in Grade 1 should be on solving a wide range of word problems that are understandable in relationship to the child's environment. By solving carefully constructed sets of problems, children can learn what addition and subtraction is and will develop procedures for solving problems. By having broad based problem solving experiences, children will be able to abstract broad based definitions of addition and subtraction. They will learn that each addition or subtraction fact can represent a large number of situations. Children will also be able to count, to recall basic facts and to do simple computation. In short, children will learn that the world is made a more orderly place through the use of mathematics; they will learn that mathematics is useful and understandable; and they will learn the skills of using addition and subtraction.

The content of a cognitively guided addition/subtraction curriculum has three facets: problem solving, processes and understandings. Each facet will be discussed briefly.

Problem Solving

No one needs to be convinced that a major goal of mathematics instruction is the improvement of problem solving skills. Humans must be able to use the mathematics they have learned in a variety of ways. While mathematics may be important to some people just because it is mathematics, most people increasingly need to be able to use mathematics to solve real problems. Of course, these real problems vary in complexity from relatively simple solutions (balancing a checkbook) to highly complex situations (putting a person on the moon).

Counting is probably the first mathematics that children use to solve problems. When children enter school many -- if not most -- have acquired reasonably sophisticated processes (most of which involve counting) for analyzing and solving certain kinds of addition and subtraction word problems. However, the traditional curriculum does not build systematically on this knowledge or support its development. As a consequence, children often do not see the connection between their informal knowledge and the formal mathematics they are taught in school. As a result, children's problem solving skills, which they have developed intuitively before formal instruction, often deteriorate during early elementary years.

Traditional curriculum has been based on the assumption that computation skills must be learned before children are taught to solve even simple word problems, Word problems have often appeared only at the end of a unit, emphasizing the computation skills taught in the unit. Whatever value such exercises may have for practicing the computational skill, they do very little to teach problem solving. Because the students are often taught that they are going to subtract the smaller number from the larger in each problem, such exercises encourage children to perform rote computations and not to read carefully or to think about problems, Thus, these exercises may actually contribute to the deterioration of children's problem solving skills. Furthermore, the exercises often provide a limited exposure to different types of addition and subtraction problems. The research summarized above shows clearly that, even before they receive formal instruction in addition and subtraction, young children can solve a wide variety of addition and subtraction problems by modeling the problems with manipulatives or by using different counting strategies. Based on this research, it appears that word problems are appropriate for introducing addition and subtraction and that word problems cannot only be integrated into the mathematics curriculum but should form the basis of it.

Processes

The acquisition of certain basic mathematical processes should also be an important goal of a mathematics program. Some of the important processes that first grade children should learn are counting in a variety of ways, representing word problems and recalling basic facts. Each of these processes is essential if one is to solve problems and to learn more sophisticated mathematics.

The process of representing a real world or word problem is fundamental to the ability to solve the problem. This representation can be in at least two forms: direct physical modeling (the use of objects of fingers) and/or use of traditionally accepted symbolism. Young children have little trouble with direct modeling -- but there is evidence that symbolizing gives major difficulty to many children. In fact, we believe that if children do not learn to symbolize many types of addition/subtraction word problems, they will experience major difficulties in learning mathematics.

Acquisition of counting processes is also a major goal of mathematics instruction. Not only should children learn to count by ones (starting at one), they should also learn to count on from any given number, count back from a given number, and to count by twos, threes and fives. While the learning of the number names in sequence must be done without much understanding, real counting must involve understanding.

Another major process which should be a goal of any mathematics program is the recalling of number facts. Although many addition and subtraction problems can be solved by counting, this process is inefficient and cumbersome. Learning addition and subtraction number facts at a recall level continues to be an important goal of the mathematics curriculum.

Understandings

We believe that the way to accomplish the goals of the mathematics curriculum is for each child to understand what is taught. While the recall of number facts, the counting processes and the representation processes should at some time become automatic, the only way to ensure that children can use those processes, is for each child to learn the interrelationships between and within the various processes. Only when each child understands can the goals of mathematics instruction be achieved.

How Children Solve Simple Word Problems

It is commonly assumed that word problems are difficult for children at all levels of the mathematics curriculum. However, there is a growing body of research clearly showing that young children are very good at solving simple word problems and their solutions are often creative and involve relative sophisticated problem-solving processes. Children enter school with quite well-developed informal systems of mathematics. Some of the mathematics they have learned through interaction with parents or siblings or from programs like "Sesame Street," but much of it they have invented for themselves. Children's solutions to simple word problems appear to be their own invention rather than the result of instruction.

Solving Addition Problems

Even before children receive any formal instruction in addition and subtraction, they can solve simple addition and subtraction word problems. Although they many not have any formal knowledge of addition or subtraction, most school age children can count, and they solve simple addition and subtraction problems by modeling the action in the problem using physical objects or fingers and counting. For example, to solve the following problem, a young child would make a set of 5 objects and another set of 8 objects, put the two sets together, and count the objects in the resulting set.

> *Tom had 5 marbles. He won 8 more marbles from his sister Connie. How many marbles does he have now?*

Younger children can solve problems like this if they have concrete objects to physically model the problem or if the numbers in the problem are small enough that they can use their fingers. Older children invent more efficient counting procedures. One of these procedures is called "counting on". To solve the above problem by counting on from the first number, a child recognized that it is not necessary to reconstruct the entire counting sequence. The child begins counting with 5 and counts on 8 more counts: "5 (pause), 6, 7, 8, 9, 10, 11, 12, 13. The answer is 13." In this procedure, the child is not actually physically modeling the problem but is abstracting the modeling process with the counting sequence. At the same time, the child is curtailing the process, just using the part of the counting sequence that is necessary to generate the answer.

A slightly more sophisticated and efficient version of counting on is called "counting on from larger." In this case, the counting sequence begins with the larger of the two numbers. "8 (pause), 9, 10, 11, 12, 13."

Both counting-on strategies require that children keep track of the number of steps in the counting sequence in order to know when to stop counting. In other words, they not only count from 5 or 8 to 13, they must count each of the numbers in the counting sequence. This involves some sort of double counting. Most children keep track on their fingers, extending a finger for each count, but some actually keep track in their heads.

Solving Subtraction Problems

Children also solve subtraction word problems by modeling the action or relationships described in the problem. This results in very different solutions for different types of problems. Although there are distinctions between different types of addition problems, all the simple addition problems that young children can readily solve can be reasonably modeled by the process described above.

For subtraction there are several distinct strategies that children use reflecting basic differences in the structure of the problem.

The four subtraction problems in Table 1 illustrate basic distinctions between problem types. Although all four problems can be represented by the mathematical sentence 13 - 5 = ?, they present distinct interpretations of subtraction.

Table 1

Classes of Subtraction Word Problems

Problem Type	Example Problem
Separate (Result Unknown)	Tom has 13 marbles. He gave 5 marbles to his sister Connie. How many marbles does Tom have left?
Join (Change Unknown)	Tom has 5 marbles. His sister Connie gave him some more marbles. Now Tom has 13 marbles. How many marbles did Connie give him?
Compare (Difference Unknown)	Tom had 5 marbles. His sister Connie had 13 marbles. How many more marbles does Connie have than Tom?
Part-Part-Whole (Part Unknown)	There are 13 marbles in the bag. Five of them are red and the rest are blue. How many blue marbles are in the bag?

The first problem describes the action of removing a subset of a given set. The second, describing an additive change action, has as its unknown the size of that change. The third problem involves the comparisons of two distinct sets, and the fourth is a static situation in which one of the two parts of a known whole must be found. Altogether there are six basic semantic types. For each semantic problem type, three distinct problems can be generated by varying which quantity is unknown. For example, the Compare problem above could be altered as follows to produce a parallel addition problem:

Tom has some candies. He gave 5 candies to his sister Connie.
If he has 7 candies left, how many did he have to start with?

As with addition problems, the youngest children attempt to represent directly the action or relationship described in the problem using physical objects or fingers. To solve the Separate problem given above, they make a set of 13 objects, remove 5, and count the remaining objects to find the answer. On the other hand, the Join (Change Unknown) problem is solved by initially taking a set of 5 objects and incrementing it until there is a total of 13 objects. The answer is found by counting the objects added to the initial set. The compare problem generates a third method of solution. In this case, children tend to make two sets, put them in one-to-one correspondence, and count the unmatched objects.

Children eventually solve subtraction problems using counting strategies that are similar to the counting strategies described for addition problems. To solve the Join (Change Unknown) problem, a child recognizes that it is not necessary to construct the initial set of 5 objects and simply starts counting at 5 and counts to 13. In this case, the answer is the number of steps in the counting sequence rather than the final number. The Separate problem is solved by counting backwards. For the problem in Table 1, counting sequence would be, "13, 12, 11, 10, 9, 8." There is no parallel counting strategy for the matching strategy used to solve the Compare problem.

Levels of Development of Problem Solving

Children's ability to solve simple addition and subtraction problems develops through four general levels.

In the earliest level, children can only solve problems that they can model directly with objects. They are unable to solve a problem like the following in which the initial quantity is unknown, as it is impossible to represent the action in the problem without trial and error.

John had some marbles. He lost 8 of them. Then he had 5 left.
How many marbles did he have to start with?

Children progress from this initial level along two dimensions: They begin to adopt more abstract versions of their modeling and they become more able to choose strategies that are not direct representations of the structure of the problem. The second level is a transition period. At this level, children use both modeling and counting strategies. Most children at this level continue to reflect the structure of the problem in their solution, whether they use counting or modeling with physical objects.

In level three, children rely primarily on counting strategies--counting on to solve addition problems, and counting down or counting up to solve subtraction problems. Although children at this level count down, most of them do not do so consistently. Level 3 children are more flexible in their ability to choose strategies and can select strategies that do not directly reflect problem structure. Therefore, children at this level can count up to solve any subtraction problem and most of them choose this strategy rather than counting down.

In the final level, children rely primarily on number facts. But number facts are not learned all at once and many children use both counting strategies and number facts concurrently. Children also use a core of recalled number facts to generate other facts. Usually these solutions are based on doubles or pairs that sum to 10. For example, to solve a problem like 6 + 8, a student would respond that you take one from the 8 and put it with the 6, so the answer is the same as 7 + 7.

The evidence suggests that most children pass through all four of the levels described above, although performance within each level may vary and the age at which a given level is attained and the duration of the levels also is different for different children. Most beginning first graders are in level 1, although some have passed on to more advanced levels. By the end of first grade, most children are in level 2 or 3. Children do not completely master number facts for an extended period of time and the extensive use of counting strategies has been observed as late as the seventh grade. It should also be noted that children become extremely proficient at counting, and may be using counting strategies when it appears that they have mastered number facts at a recall level.

Implications for Instruction

The research on addition and subtraction clearly suggests that the current primary mathematics curriculum fails to capitalize on the rich informal mathematics that children bring to instruction. Children's invented strategies for solving addition and subtraction problems are frequently more efficient and more conceptually based than mechanical procedures included in many mathematics programs. How instruction should be designed to incorporate these findings is a complex issue. There are a number of ways that instruction might be sequenced to build upon our knowledge of how addition and subtraction concepts develop naturally in children. Perhaps the most obvious is to pattern instruction along the lines of the observed level of development. In other words, lead children through the levels by explicitly teaching the strategies of each succeeding level. At earliest levels, instruction would be based on directly modeling strategies and problems would be restricted to those that could be represented directly using physical objects. The next level of instruction would focus on the more abstract counting strategies which would be succeeded by the learning of number facts at a recall level.

There has been some success in explicitly teaching strategies in which children use a core of recalled facts to derive additional facts. The instruction has usually been built upon children's natural knowledge of doubles (6 + 6) and other key sets of facts. Other facts are related to this core of known facts (6 + 7 = 6 + 6 + 1).

The research on addition and subtraction suggests some additional possibilities for sequencing instruction and emphasizing specific concepts or approaches. One possible direction is that basic addition and subtraction word problems be used as a basis for developing addition and subtraction concepts rather than teaching the computational skills first and then applying them to solve problems.

A second potential implication regards the use of on canonical open sentences like 8 + ? = 13. Such problems were widely used in many programs in the 1960s, but their use has declined. These sentences, however, provide the best symbolic representation of the processes that children naturally use to solve Join (Change Unknown) problems (see Table 1). Some preliminary research suggests that instruction on noncononical number sentences enhances children's ability to mathematically represent and solve a variety of different types of problems.

Perhaps the most immediate implication of the research on addition and subtraction word problems is that it gives teachers a framework for understanding their own student's thought processes. Most of the results discussed above come from clinical interviews in which children were asked to solve problems and explain their solutions. By spending just a little time talking to a child about his or her solution to a problem, a teacher can gain a great deal of insight about the child's knowledge of mathematics and problem-solving strategies.