Math 535, Summer 2014: Looking for Number Patterns

Starting with the ancient Greeks, mathematicians asked questions about patterns of whole numbers. Think of it as trying to solve very hard puzzles. Here are some examples.

1. Twin primes: Euclid knew that there are infinitely many prime numbers and noticed that some pairs of consecutive odd numbers are both primes

examples: (3,5), (5,7), not(7,9), give other examples

Such pairs of consecutive odd numbers with both being prime are called "twin primes".

Question: Why don't we ask about consecutive even numbers that are both prime?

How many twin primes are there? infinitely many? or only a finite number? Nobody knows, but there has been some recent progress on a related problem (see item (3) below).

2. Perfect numbers: The Greeks knew that some numbers are the sum of their proper divisors. For example, $6 = 1 + 2 + 3$. Such numbers are called "perfect numbers".

What about 10?

What about 28?

Are there infinitely many perfect numbers? Nobody knows. Are there any odd perfect numbers? Nobody knows.

3. Gaps between primes: In our number line experiment, we saw that there are lots of primes early, and fewer primes later. How long can the gaps between primes be? Bertrand's Theorem says that between any whole number and its double (i.e., in every interval $[n, 2n]$) there is at least one prime. In 2013, a mathematician proved that there are infinitely many pairs of prime numbers whose difference is less than 70 million. More rencetly, mathematicians have proved that there are infinitely many pairs of primes whose difference is less than 600. Follow this developing story by looking up "prime pairs" on the Internet.

Question: How is this research on (3) related to the Twin Prime problem mentioned in (1)?

4. Sums of primes: The Goldbach conjecture says "Every even integer greater than 2 is the sum of 2 primes". This has been checked by computer out to 4×10^{18} and that provides evidence that Goldbach's conjecture is true. But nobody knows.

Question: Why does checking out to 4×10^{18} not prove the conjecture?

The LeMoine conjecture is that every odd number greater than 5 can be written as $p + 2q$ where p and q are primes.

(Example: $11 = 5 + (2 \times 3)$, $9 = 3 + (2 \times 3), 13 =$)

Nobody knows whether Goldbach's conjecture or LeMoine's conjecture is true.

5. Sums of other interesting number patterns: Lagrange proved that every positive integer is the sums of at most four perfect squares, e.g., $2 = 1 + 1, 17 = 1^2 + 4^2, 30 = 1 + 4 + 9 + 16$.

Express 17 as the sums of up to four perfect squares in another way.

Show that 15 cannot be written as the sum of three perfect squares.

6, The $3n+1$ problem. Here is a game. Start with any whole number, say n. If n is even, replace it by $\frac{n}{2}$. If n is odd, replace it by $3n+1$. Repeat over and over. Conjecture: You always end up with the number 1 after several steps.

Examples: $5 \to 16 \to 8 \to 4 \to 2 \to 1$

$7 \to 22 \to 11 \to \cdots$

$24 \to 12 \to 6 \to \cdots$

Nobody knows whether this game always ends up with the number 1.

Hailstone Numbers

by Ivars Peterson

Nothing could be grayer, more predictable, or less surprising than the endless sequence of whole numbers. Right? That's why people count to calm down and count to put themselves to sleep. Whole numbers define boooooooring.

Not so fast. Many mathematicians like playing with numbers, and sometimes they discover weird patterns that are hard to explain. Here's a mysterious one you can try on your calculator.

Pick any whole number. If it's odd, multiply the number by 3, then add 1. If it's even, divide it by 2. Now, apply the same rules to the answer that you just obtained. Do this over and over again, applying the rules to each new answer.

For example, suppose you start with 5. The number 5 is odd, so you multiply it by 3 to get 15, and add 1 to get 16. Because 16 is even, you divide it by 2 to get 8. Then you get 4, then 2, then 1, and so on. The final three numbers keep repeating.

Try it with another number. If you start with 11, you would get 34, 17, 52, 26, 13, 40, 20, 10, 5, 16, 8, 4, 2, 1, and so on. You eventually end up at the same set of repeating numbers: 4, 2, 1. Amazing!

The numbers generated by these rules are sometimes called "hailstone numbers" because their values go up and down wildly—as if, like growing hailstones, they were being tossed around in stormy air—before crashing to the ground as the repeating string 4, 2, 1.

Mathematicians have tried every whole number up to at least a billion times a billion, and it works every time. Sometimes it takes only a few steps to reach 4, 2, 1; sometimes it takes a huge number of steps to get there. But you get there every time.

Does that mean it would work for any whole number you can think of—no matter how big? No one knows for sure. Just because it works for every number we've tried doesn't guarantee that it would work for all numbers. In fact, mathematicians have spent weeks and weeks trying to prove that there are no exceptions, but they haven't succeeded yet. Why this number pattern keeps popping up remains a mystery.

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